

SUPERSYMMETRY AND GAUGE THEORY IN CALABI–YAU 3-FOLDS

JM FIGUEROA-O’FARRILL, A IMAANPUR, AND J MCCARTHY

ABSTRACT. We consider the dimensional reduction of supersymmetric Yang–Mills on a Calabi–Yau 3-fold. We show by construction how the resulting cohomological theory is related to the balanced field theory of the Kähler Yang–Mills equations introduced by Donaldson and Uhlenbeck–Yau.

1. INTRODUCTION

The study of Ricci-flat manifolds is interesting to both geometers and string theorists for a variety of reasons. These manifolds provide examples of “exotic” Einstein geometries: in fact, their holonomy groups have to be $SU(n)$, $Sp(n)$, G_2 or $Spin(7)$, corresponding to Calabi–Yau n -folds, hyperkähler manifolds of real dimension $4n$, and exceptional 7- and 8-manifolds, respectively. Because they are Ricci-flat and admit parallel spinors, they are supersymmetric vacua for superstring-related theories. Out of these parallel spinors one can construct parallel forms [19, 15] which turn out to be calibrations in the sense of [16]. Indeed, these manifolds have a rich geometry of (calibrated) minimal submanifolds. These submanifolds are, in the simplest case, the supersymmetric cycles [5] around which branes may wrap to produce BPS states. Yang–Mills theory on these manifolds is also interesting. The equations of motion admit instantonic solutions which minimise the action and are defined by linear equations generalising (anti)self-duality in four dimensions [11, 20]. This observation forms the basis of the “Oxford programme” [14] to generalise Donaldson–Floer–Witten theory to higher dimensional Ricci-flat manifolds.

Perhaps one of the boldest proposals yet to have emerged out of the “second superstring revolution” is the Matrix Conjecture of Banks *et al.* [3]. This conjecture states that the dimensional reduction to one dimension of 10-dimensional supersymmetric Yang–Mills in the limit in which the rank of the gauge group goes to infinity provides an Infinite Momentum Frame description of M-theory, the 11-dimensional theory believed to underlie nonperturbative superstring theory. In this context, it becomes an important problem to understand the

Supported by the UK EPSRC under contract GR/K57824.

Supported by the Ministry of Culture and Higher Education, Iran.

Supported by the Australian Research Council.

dimensional reductions on 10-dimensional supersymmetric Yang–Mills theory. Most research has focused on toroidal compactifications, since these preserve all of the sixteen supercharges present in the original theory, and are therefore the most constrained. On the other hand, reductions on curved Ricci-flat manifolds, also produce manageable theories even though there is little supersymmetry left. The reason is that, as we will review below, whatever supersymmetry remains becomes BRST-like, rendering the theory cohomological.

The theory we will describe in what follows can be understood as that arising out of euclidean D-branes wrapping around a Calabi–Yau 3-fold. More prosaically, it is the dimensional reduction of 10-dimensional supersymmetric Yang–Mills theory to such a manifold. Results in this direction for other manifolds have been obtained in [7, 10], who considered euclidean D-branes wrapping around calibrated submanifolds. The resulting theories on the D-brane were seen to be topologically twisted Yang–Mills theory – the components of the 10-dimensional gauge field in directions normal to the D-brane being sections of the normal bundle to the calibrated submanifold which need not be trivial. In [1] the dimensional reductions of supersymmetric Yang–Mills to 7- and 8-manifolds of exceptional holonomy (G_2 and $\text{Spin}(7)$, respectively) were studied. The theories obtained are cohomological [22] and localise on the moduli space of generalised instantons and, in the 7-dimensional case, monopoles. The instanton theories agree (morally) with the cohomological theories studied in [4, 2]. Similar considerations, in less detail but in more generality, can be found in [8].

In this paper we will follow the approach of [1] and study the theory on a Calabi–Yau 3-fold. We will recover a cohomological theory which localises on the moduli space of solutions of the Kähler–Yang–Mills equations. These equations have been studied by Donaldson [13] (for Kähler surfaces) and by Uhlenbeck–Yau [18] (in complex dimension three and above), who show that they are in one-to-one correspondence with stable holomorphic vector bundles. Cohomological theories which localise on this moduli space have been discussed in [4] as a reduction of the eight-dimensional cohomological theories, and also briefly in [8].

Our approach will be the following. We start with 10-dimensional supersymmetric Yang–Mills theory and reduce it to 6-dimensional euclidean space. The resulting lagrangian can be promoted to any spin 6-manifold M by simply covariantising the derivatives with respect to the spin connection; but the supersymmetry transformations will fail to be a symmetry of the action unless the spinorial parameters are covariantly constant. This requires that M admit parallel spinors, and that means that the holonomy group must be a subgroup of $\text{SU}(3)$. If we want M to be irreducible then the holonomy must be $\text{SU}(3)$. Covariance of the supersymmetry algebra under the holonomy group implies that the commutator of two supersymmetry transformations with

parallel spinors as parameters will result (on shell and up to gauge transformations) in a translation by a parallel vector. Since for the irreducible manifolds we consider there are no such vectors, the supersymmetry transformation is a BRST symmetry. This general argument shows that the resulting theory is cohomological.

This paper is organised as follows. In Section 2 we discuss the dimensional reduction of 10-dimensional supersymmetric Yang–Mills theory to 6-dimensional euclidean space. In Section 3 we specialise to the theory defined on a manifold of holonomy $SU(3)$: a Calabi–Yau 3-fold, and show that it is indeed cohomological. In Section 4 we rewrite the theory in the form of a balanced cohomological field theory in the sense of [12] and [9].

For convenience we briefly summarise our spinor conventions here. We use the Minkowski signature $(-1, 1, 1, \dots, 1)$. The unitary charge conjugation matrix for the Clifford algebra generators γ_μ is specified, for given $\sigma_d, \sigma_t \in \{\pm 1\}$, by

$$C\gamma_\mu C^{-1} = \sigma_d \gamma_\mu^t \quad \text{and} \quad C^t = \sigma_t C . \quad (1)$$

For the spinor representations of $SO(3, 1)$ we use notation along the lines of Wess and Bagger [21], $\sigma^I = (1, \sigma^i)$ with indices σ_{aa}^I and $\bar{\sigma}^I = (-1, \sigma^i)$ obeying $\bar{\sigma}^{I\dot{a}a} = -\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma_{bb}^I$. We choose $\epsilon^{12} = 1$ and $\epsilon^{ab}\epsilon_{bc} = -\delta_c^a$. The dot distinguishes between the two spinor representations, ψ_a and $\bar{\psi}^{\dot{a}}$, for which the $SO(3, 1)$ generators are

$$\sigma^{IJ} \equiv \frac{1}{4}(\sigma^I \bar{\sigma}^J - \sigma^J \bar{\sigma}^I) \quad \text{and} \quad \bar{\sigma}^{IJ} \equiv \frac{1}{4}(\bar{\sigma}^I \sigma^J - \bar{\sigma}^J \sigma^I) . \quad (2)$$

It is straightforward to see that with $(\psi_a)^\dagger \equiv \bar{\psi}_{\dot{a}}$ and $(\bar{\psi}^{\dot{a}})^\dagger \equiv \psi^a$, we have, e.g., $\bar{\psi}^{\dot{a}} = \epsilon^{\dot{a}b}\bar{\psi}_{\dot{b}}$.

2. DIMENSIONAL REDUCTION TO SIX DIMENSIONS

Our starting point is 10-dimensional supersymmetric Yang–Mills theory. It can be formulated in terms of a Lie algebra valued gauge field A_M and a negative chirality Majorana–Weyl adjoint spinor Ψ . The Lie algebra is assumed to possess an invariant metric, denoted $(-, -)$ or sometimes Tr . The lagrangian is then given by

$$\mathcal{L} = -\frac{1}{4}(F_{MN}, F^{MN}) + \frac{i}{2}(\bar{\Psi}, \Gamma^M D_M \Psi) , \quad (3)$$

where $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$, and $D_M \Psi = \partial_M \Psi + [A_M, \Psi]$, and

$$\bar{\Psi} = \Psi^t \mathbf{C} , \quad (\sigma_d^{(10)} = \sigma_t^{(10)} = -1) . \quad (4)$$

This action is hermitian and invariant under the following supersymmetry transformations,

$$\delta A_M = i\bar{\varepsilon}\Gamma_M \Psi \quad \text{and} \quad \delta \Psi = \frac{1}{2}F_{MN}\Gamma^{MN}\varepsilon , \quad (5)$$

where ε is a constant negative chirality Majorana–Weyl spinor. The supersymmetry algebra only closes on-shell and up to gauge transformations.

Reducing the theory down to six euclidean dimensions breaks the 10-dimensional Lorentz invariance down to a subgroup $\text{SO}(3, 1) \times \text{SO}(6)$. The first step of the dimensional reduction is then the decomposition of our fields into irreducible representations of this subgroup. For tensor fields this is the obvious decomposition $M = (I, \mu)$; in particular, $A_M = (\phi_I, A_\mu)$. For the spinors we use the representation

$$\Gamma^I = \tilde{\gamma}^I \otimes \bar{\gamma}_7 \quad \text{and} \quad \Gamma^{\mu+4} = \mathbb{1}_4 \otimes \bar{\gamma}^\mu, \quad (6)$$

where $\bar{\gamma}^\mu$ are the generators for the Clifford algebra $\text{Cl}(6, 0)$ and $\tilde{\gamma}^I$ for $\text{Cl}(3, 1)$. A straightforward calculation gives

$$\Gamma_{11} = \tilde{\gamma}_5 \otimes \bar{\gamma}_7. \quad (7)$$

The charge conjugation matrix \mathbf{C} then decomposes as $\mathbf{C} = \tilde{C} \otimes \bar{C}$. For definiteness we choose the representation in which the $\bar{\gamma}^\mu$ are all antisymmetric ($\sigma_d^{(6)} = -\sigma_t^{(6)} = -1$), and the chiral representation for the 4-dimensional γ 's ($\sigma_d^{(4)} = -\sigma_t^{(4)} = 1$); in terms of Pauli matrices,

$$\tilde{\gamma}^0 = \mathbb{1}_2 \otimes (i\sigma^2) \quad (8)$$

$$\tilde{\gamma}^i = \sigma^i \otimes \sigma^1 \quad \text{for } i = 1, 2, 3 \quad (9)$$

$$\tilde{\gamma}_5 = \mathbb{1}_2 \otimes \sigma^3, \quad (10)$$

with $\tilde{C} = i\sigma^2 \otimes \sigma^3$.

Let e_a , $a = 1, 2$ denote an orthonormal eigen-basis of σ^3 . Then the Weyl condition determines

$$\Psi = e^a \otimes e_1 \otimes \psi_{La} + e_{\dot{a}} \otimes e_2 \otimes \psi_R^{\dot{a}}. \quad (11)$$

The 10-dimensional Majorana condition then reduces to a reality condition on the 6-dimensional fields,

$$\bar{\psi}_{Lb} = -\psi_R^{t\dot{a}} \epsilon_{\dot{a}b} \quad \bar{\psi}_R^b = \psi_{La}^t \epsilon^{ab} \quad (12)$$

$$\psi_{La} = \epsilon_{ab} \psi_R^{*b} \quad \psi_R^{\dot{a}} = -\epsilon^{\dot{a}b} \psi_{Lb}^*. \quad (13)$$

Finally then, the lagrangian reduces to

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} \|F_{\mu\nu}\|^2 - \frac{1}{2} (D_\mu \phi_I, D_\mu \phi^I) - \frac{1}{4} ([\phi_I, \phi_J], [\phi^I, \phi^J]) \\ & + \frac{i}{2} (\bar{\psi}_R^a, \bar{\gamma}_\mu D_\mu \psi_{La}) + \frac{i}{2} (\bar{\psi}_{L\dot{a}}, \bar{\gamma}_\mu D_\mu \psi_R^{\dot{a}}) + i (\bar{\psi}_R^a, \sigma_{ab}^I [\phi_I, \psi_R^{\dot{b}}]) , \end{aligned} \quad (14)$$

which is invariant under the supersymmetry transformations

$$\delta \psi_{La} = \frac{1}{2} \bar{\gamma}_{\mu\nu} F_{\mu\nu} \epsilon_{La} + \bar{\gamma}_\mu D_\mu \psi_I \sigma_{ab}^I \epsilon_R^{\dot{b}} + [\phi_I, \phi_J] \sigma_a^{IJb} \epsilon_{Lb} \quad (15)$$

$$\delta \psi_R^{\dot{a}} = \frac{1}{2} \bar{\gamma}_{\mu\nu} F_{\mu\nu} \epsilon_R^{\dot{a}} - \bar{\gamma}_\mu D_\mu \psi_I \bar{\sigma}^{I\dot{a}b} \epsilon_{Lb} + [\phi_I, \phi_J] \bar{\sigma}^{IJ\dot{a}}_{\dot{b}} \epsilon_R^{\dot{b}} \quad (16)$$

$$\delta A_\mu = i \bar{\epsilon}_R^a \bar{\gamma}_\mu \psi_{La} + i \bar{\epsilon}_{L\dot{a}} \bar{\gamma}_\mu \psi_R^{\dot{b}} \quad (17)$$

$$\delta \phi_I = -i \bar{\epsilon}_{L\dot{a}} \bar{\sigma}_I^{\dot{a}b} \psi_{Lb} + i \bar{\epsilon}_R^a \sigma_{Iab} \psi_R^{\dot{b}}. \quad (18)$$

3. REDUCTION TO MANIFOLDS WITH $SU(3)$ HOLONOMY

Now that we have a supersymmetric theory defined on a six dimensional euclidean space, it is time to extend it to a Calabi–Yau 3-fold. The structure group of the tangent bundle reduces to an $SU(3)$ subgroup of $SO(6)$. Our first task is to decompose the $SO(6)$ fields into irreducible representations of $SU(3)$. We will actually consider the decomposition into $U(3)$ irreducibles, $U(3)$ being the holonomy group of a 6-dimensional Kähler manifold. Since $U(3)$ is locally isomorphic to $SU(3) \times U(1)$, we will be able to read off the $SU(3)$ representations easily. It is, of course, sufficient to work in a local frame.

The embedding $SO(6) \supset SU(3) \times U(1)$ leads to the branching $\mathbf{4} = (\mathbf{1})_3 \oplus (\mathbf{3})_{-1}$; thus, under the global symmetry $SU(3) \times U(1) \times SO(3, 1)$, we have that the spinors λ_R and λ_L transform according to

$$\lambda_R \sim (\mathbf{1}, 3, \mathbf{2}_L) \oplus (\mathbf{3}, -1, \mathbf{2}_L) \quad (19)$$

$$\lambda_L \sim (\mathbf{1}, -3, \mathbf{2}_R) \oplus (\bar{\mathbf{3}}, 1, \mathbf{2}_R) . \quad (20)$$

Let θ denote the (commuting, left-handed) spinor which is responsible for splitting the $\mathbf{4}$ above, and let us normalise it to $\theta^\dagger \theta = 1$. Clearly θ^* is the right-handed singlet spinor, which splits the $\bar{\mathbf{4}}$. We need the explicit projections onto these representations. The projector onto the singlet in the $\bar{\mathbf{4}}$ is

$$\theta \theta^\dagger = \frac{1}{8}(1 - \bar{\gamma}_7) - \frac{1}{16} \theta^\dagger \bar{\gamma}_{\mu\nu} \theta (1 - \bar{\gamma}_7) \bar{\gamma}_{\mu\nu} , \quad (21)$$

as follows from the standard result

$$\bar{\gamma}^{\mu_1 \dots \mu_r} = (-1)^{1+r(r-1)/2} \frac{i}{(6-r)!} \epsilon^{\mu_1 \dots \mu_6} \bar{\gamma}_7 \bar{\gamma}_{\mu_{r+1} \dots \mu_6} , \quad (22)$$

together with a Fierz transformation upon noticing that by chirality

$$\theta^\dagger \bar{\gamma}^{(A)} \theta = 0 , \quad \text{for } |A| \text{ odd.} \quad (23)$$

It follows immediately from

$$\bar{\gamma}^\lambda \bar{\gamma}_{(A)} \bar{\gamma}_\lambda = (-1)^{|A|} (6 - 2|A|) \bar{\gamma}_{(A)} \quad (24)$$

that

$$\bar{\gamma}_\lambda \theta \theta^\dagger \bar{\gamma}_\lambda = \frac{3}{4} (1 + \bar{\gamma}_7) - \frac{1}{8} \theta^\dagger \bar{\gamma}_{\mu\nu} \theta (1 + \bar{\gamma}_7) \bar{\gamma}_{\mu\nu} , \quad (25)$$

which will also be required later.

We may now introduce the Kähler form $k_{\mu\nu} \equiv i \theta^\dagger \bar{\gamma}_{\mu\nu} \theta$ and the 3-form

$$\Omega_{\mu\nu\lambda} \equiv \theta^\dagger \bar{\gamma}_{\mu\nu\lambda} \theta^* . \quad (26)$$

There are no other covariants since

$$\theta^\dagger \bar{\gamma}^\mu \theta^* = 0 , \quad (27)$$

which follows from sandwiching (the complex conjugate of) (25) between θ^t and θ^* .

At this point it is useful to make a special choice of θ which corresponds to the standard choice of complex coordinates. This reduces the

problem to the usual construction of the spinor representation of $SO(6)$ via linear combinations of the Clifford algebra generators which obey the algebra of fermionic oscillators. First introduce the combinations (taking $\mu = (\alpha, \bar{\alpha})$ in flat (local frame) coordinates)

$$\gamma^\alpha = \frac{1}{\sqrt{2}}(\bar{\gamma}^\alpha + i\bar{\gamma}^{\alpha+3}) \quad \text{and} \quad \gamma^{\bar{\alpha}} = \frac{1}{\sqrt{2}}(\bar{\gamma}^\alpha - i\bar{\gamma}^{\alpha+3}) . \quad (28)$$

The $SU(3)$ generators are then $T^\alpha{}_\beta = \gamma^\alpha \gamma_\beta - \frac{1}{3} \delta^\alpha_\beta \gamma^\gamma \gamma_\gamma$. Requiring that θ be an $SU(3)$ singlet (with appropriate $U(1)$ charge -3) fixes $\gamma_\alpha \theta = 0$, so that $k_{\alpha\beta} = k_{\bar{\alpha}\bar{\beta}} = 0$ and $k_{\alpha\bar{\beta}} = i\delta_{\alpha\bar{\beta}}$. Similarly all components of Ω vanish by (27) but for $\Omega_{\alpha\beta\gamma} \equiv \theta^\dagger \gamma_{\alpha\beta\gamma} \theta^*$ and its conjugate. For completeness, notice that

$$\Omega_{\alpha\beta\gamma} \bar{\Omega}_{\bar{\alpha}\bar{\beta}'\bar{\gamma}'} = 8(\delta_{\beta\bar{\beta}'} \delta_{\gamma\bar{\gamma}'} - \delta_{\beta\bar{\gamma}'} \delta_{\gamma\bar{\beta}'}) . \quad (29)$$

Using vielbeins to translate to the coordinate basis, these results apply for an arbitrary Kähler (6d) manifold.

The projectors for spinors onto $SU(3) \times U(1)$ covariant fields follow directly by combining (21) and (the complex conjugate of) (25) to get the appropriate completeness relations; e.g.,

$$\frac{1}{2}(1 - \bar{\gamma}_7) = \theta\theta^\dagger + \frac{1}{2}\gamma_\alpha \theta^* \theta^t \gamma_{\bar{\alpha}} . \quad (30)$$

For arbitrary symplectic Majorana–Weyl spinors χ_L or χ_R , define

$$\chi_a = \theta^\dagger \chi_{La} \quad \text{and} \quad \chi_{a\bar{\alpha}} = \theta^t \gamma_{\bar{\alpha}} \chi_{La} . \quad (31)$$

Then the $SO(3,1)$ covariant decompositions under $SU(3) \times U(1)$ are

$$\chi_{La} = \theta \chi_a + \frac{1}{2} \gamma_\alpha \theta^* \chi_{a\bar{\alpha}} \quad \text{and} \quad \chi_R^{\dot{a}} = -\theta^* \epsilon^{\dot{a}\dot{b}} \bar{\chi}_{\dot{b}} + \frac{1}{2} \gamma_{\bar{\alpha}} \theta \epsilon^{\dot{a}\dot{b}} \bar{\chi}_{b\alpha} , \quad (32)$$

and in terms of these fields, the lagrangian becomes

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(F_{\alpha\beta}, F_{\bar{\alpha}\bar{\beta}}) - \frac{1}{2}(F_{\alpha\bar{\beta}}, F_{\bar{\alpha}\beta}) - (D_\alpha \phi^I, D_{\bar{\alpha}} \phi_I) \\ & - \frac{1}{4}([\phi_I, \phi_J], [\phi^I, \phi^J]) + i\epsilon^{ab}(\psi_a, D_\alpha \psi_{b\bar{\alpha}}) + i\epsilon^{\dot{a}\dot{b}}(\bar{\psi}_{\dot{a}}, D_{\bar{\alpha}} \bar{\psi}_{b\bar{\alpha}}) \\ & - i\bar{\sigma}^{I\dot{a}\dot{b}}(\bar{\psi}_{\dot{a}}, [\phi_I, \psi_b]) - \frac{i}{2}\bar{\sigma}^{I\dot{a}\dot{b}}(\bar{\psi}_{\dot{a}\alpha}, [\phi_I, \psi_{b\bar{\alpha}}]) \\ & - \frac{i}{8}\Omega_{\alpha\beta\gamma}\epsilon^{ab}(\psi_{a\bar{\alpha}}, D_{\bar{\beta}} \psi_{b\bar{\gamma}}) - \frac{i}{8}\bar{\Omega}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\epsilon^{\dot{a}\dot{b}}(\bar{\psi}_{\dot{a}\alpha}, D_{\beta} \bar{\psi}_{b\bar{\gamma}}) . \end{aligned} \quad (33)$$

This action is invariant with respect to the supersymmetry transformations with the parallel spinors as parameter. These are obtained from δ_S just by inserting the $SU(3)$ singlet grassmann parameters,

$$\epsilon_{La} = \theta \epsilon_a , \quad \epsilon_R^{\dot{a}} = -\theta^* \epsilon^{\dot{a}\dot{b}} \bar{\epsilon}_{\dot{b}} . \quad (34)$$

In writing the explicit supersymmetries it is convenient to introduce auxiliary fields so that the supersymmetry algebra closes off shell. Because of $SU(3)$ covariance and the fact that there are no $SU(3)$ invariant vectors on a Calabi–Yau 3-fold, the supersymmetry algebra will be

BRST-like, at least up to gauge transformations. Further, it is convenient to split the conjugate generators using the complex structure. Thereto, we introduce supercharges via

$$\delta_S = i\bar{\epsilon}_{\dot{a}}\bar{Q}^{\dot{a}} - i\epsilon_a Q^a . \quad (35)$$

The sign is such that Q and \bar{Q} act like canonical generators (so $QA = B \Rightarrow \bar{Q}A^\dagger = -B^\dagger$ and $QB = A \Rightarrow \bar{Q}B^\dagger = A^\dagger$, if A is bosonic).

The algebra of charges on the fields is then easily found to be

$$\{Q^a, Q^b\} = 0 , \quad \{Q^{\dot{a}}, Q^{\dot{b}}\} = 0 \quad \text{and} \quad \{Q^a, \bar{Q}^{\dot{b}}\} = \delta_G(-2i\bar{\sigma}^{Iba}\phi_I) , \quad (36)$$

where $\delta_G(\theta)$ means “gauge transformation with parameter θ ”. Then δ_S can be extended to the auxiliary fields

$$H = -iF_{\alpha\bar{\alpha}} \quad \text{and} \quad H_\alpha = \frac{i}{2}\Omega_{\alpha\beta\gamma}F_{\bar{\beta}\bar{\gamma}} , \quad (37)$$

so that the supersymmetry algebra is maintained off shell. We have the explicit transformations:

Field	$\bar{Q}^{\dot{a}}$	Q^a
ψ_b	0	$H\delta_b^a + i\sigma^{IJa}_b[\phi_I, \phi_J]$
$\bar{\psi}_{\dot{b}}$	$H\delta_{\dot{b}}^{\dot{a}} + i\bar{\sigma}^{IJ\dot{a}}_{\dot{b}}[\phi_I, \phi_J]$	0
$\psi_{b\bar{\alpha}}$	$2iD_{\bar{\alpha}}\phi_I\bar{\sigma}^{I\dot{a}a}\epsilon_{ab}$	$H_{\bar{\alpha}}\delta_b^a$
$\bar{\psi}_{\dot{b}\alpha}$	$H_\alpha\delta_{\dot{b}}^{\dot{a}}$	$2iD_\alpha\phi_I\epsilon^{ab}\sigma_{bb}^I$
A_α	$\epsilon^{\dot{a}b}\bar{\psi}_{b\alpha}$	0
$A_{\bar{\alpha}}$	0	$-\epsilon^{ab}\psi_{b\bar{\alpha}}$
ϕ_I	$-\bar{\sigma}_I^{\dot{a}b}\psi_b$	$\bar{\sigma}_I^{\dot{b}a}\bar{\psi}_{\dot{b}}$
H	$i\bar{\sigma}^{I\dot{a}b}[\phi_I, \psi_b]$	$i\bar{\sigma}^{Iba}[\phi_I, \bar{\psi}_{\dot{b}}]$
H_α	0	$4i\epsilon^{ab}D_\alpha\psi_b + 2i\bar{\sigma}^{Iba}[\phi_I, \bar{\psi}_{\dot{b}\alpha}]$
$H_{\bar{\alpha}}$	$4i\epsilon^{\dot{a}b}D_{\bar{\alpha}}\bar{\psi}_{\dot{b}} + 2i\bar{\sigma}^{I\dot{a}b}[\phi_I, \psi_{b\bar{\alpha}}]$	0

It is possible now to reduce to a cohomological theory with a single cohomological symmetry: setting $\psi_2 = \bar{\psi}_2 = 0$, which requires $\phi_1 = \phi_2 = 0$, we are left with the supersymmetries generated by Q^1 and $\bar{Q}^{\dot{1}}$. Instead we will keep all supersymmetries and work out a balanced formulation for this cohomological theory.

4. A BALANCED COHOMOLOGICAL FIELD THEORY

In order to recognise what this theory computes, it will prove convenient to rewrite it in balanced form [9, 12]; that is, in terms of potentials. Let us first write the lagrangian in a form linear in Q 's. To this effect, introduce

$$\bar{\mathcal{L}} = Q^a\mathcal{V}_a + \bar{Q}^{\dot{a}}\bar{\mathcal{V}}_{\dot{a}} , \quad (38)$$

where \mathcal{V}_a is dimension $\frac{7}{2}$ in the natural units where the gauge coupling is scaled out, A_μ and ϕ_I have dimension 1 and ψ 's have dimension $\frac{3}{2}$.

Further, it should be gauge invariant and an $\mathrm{SO}(3, 1)$ doublet. Taking the most general possible Ansatz and comparing to (33), we find

$$\begin{aligned} \mathcal{V}_a = & \frac{i}{4}(\psi_a, F_{\alpha\bar{\alpha}}) + \frac{1}{8}(\psi_a, H) - \frac{i}{8}\sigma^{IJb}_a(\psi_b, [\phi_I, \phi_J]) - \frac{i}{16}\Omega_{\alpha\beta\gamma}(\psi_{a\bar{\alpha}}, F_{\bar{\beta}\bar{\gamma}}) \\ & + \frac{1}{16}(\psi_{a\bar{\alpha}}, H_\alpha) - \frac{i}{8}\epsilon^{\dot{a}\dot{b}}\sigma^I_{a\dot{a}}(\bar{\psi}_{b\alpha}, D_{\bar{\alpha}}\phi_I) . \end{aligned} \quad (39)$$

Eliminating the auxiliary fields (which are determined correctly), we find that

$$\mathcal{L} = \bar{\mathcal{L}} + \frac{1}{2}(F_{\alpha\beta}, F_{\bar{\alpha}\bar{\beta}}) - \frac{1}{2}(F_{\alpha\bar{\beta}}, F_{\bar{\alpha}\beta}) - \frac{1}{2}(F_{\alpha\bar{\alpha}}, F_{\beta\bar{\beta}}) . \quad (40)$$

Note that the extra terms can be rewritten as $-\frac{1}{2}k \wedge \mathrm{Tr}(F \wedge F)$, whence their integral only depends on the Kähler class and the characteristic class of the gauge bundle.

We can pursue this a little further, writing $\bar{\mathcal{L}}$ quadratic in Q 's. The most general form is

$$\bar{\mathcal{L}} = \epsilon_{ab}Q^aQ^b\mathcal{V} + Q^a\sigma^I_{a\dot{a}}\bar{Q}^{\dot{a}}\mathcal{V}_I + \text{h.c.} , \quad (41)$$

and a similar analysis to the above gives

$$\mathcal{V}_a = \epsilon_{ab}Q^b\mathcal{V} + \sigma^I_{a\dot{a}}\bar{Q}^{\dot{a}}\mathcal{V}_I , \quad (42)$$

where

$$\mathcal{V} = -\frac{i}{32}\Omega_{\alpha\beta\gamma}\mathrm{CS}(A)_{\bar{\alpha}\bar{\beta}\bar{\gamma}} - \frac{1}{16}\epsilon^{cd}(\psi_c, \psi_d) \quad (43)$$

$$\mathcal{V}_I = -\frac{i}{16}(\phi_I, F_{\alpha\bar{\alpha}}) + \frac{1}{64}\bar{\sigma}^{\dot{b}b}_I(\bar{\psi}_{b\alpha}, \psi_{b\bar{\alpha}}) , \quad (44)$$

with $\mathrm{CS}(A)$ the holomorphic Chern–Simons 3-form,

$$\mathrm{CS}(A)_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = (A_{\bar{\alpha}}, F_{\bar{\beta}\bar{\gamma}}) - \frac{1}{3}(A_{\bar{\alpha}}, [A_{\bar{\beta}}, A_{\bar{\gamma}}]) . \quad (45)$$

Clearly

$$\bar{Q}^{\dot{a}}\mathcal{V} = 0 , \quad (46)$$

and \mathcal{V}_I is real. Thus we can write

$$\bar{\mathcal{L}} \equiv \bar{\mathcal{L}}_1 + \bar{\mathcal{L}}_2 = (\epsilon_{ab}Q^aQ^b - \epsilon_{\dot{a}\dot{b}}\bar{Q}^{\dot{a}}\bar{Q}^{\dot{b}})(\mathcal{V} + \bar{\mathcal{V}}) + 2Q^a\sigma^I_{a\dot{a}}\bar{Q}^{\dot{a}}\mathcal{V}_I \quad (47)$$

Note that the holomorphic Chern–Simons term cannot be reproduced as a BRST variation, so it isn't profitable to continue this process. That such a term—only invariant under small gauge transformations—should appear at all is quite interesting, and consistent with the results in [1] for manifolds of G_2 holonomy.

We would like to rewrite this in balanced form along the lines of [12] (see also [9]). To do that, one must first choose a global $\mathrm{SL}(2, \mathbb{R})$ under which the balanced supercharges will transform as a doublet d . The lagrangian must then be written (up to a topological term) in the form

$$\epsilon_{AB}d^Ad^B\mathcal{W} \quad \text{for some potential } \mathcal{W}, \quad (48)$$

where the critical points of \mathcal{W} agree with the fixed points of the cohomological symmetry.

It is natural, in our case, to take the $\text{SO}(2, 1)$ subgroup of the global $\text{SO}(3, 1)$ symmetry. Then the doublet supercharges can be taken as the linear combinations d^A and \tilde{d}^A , where

$$d = \begin{pmatrix} \bar{Q}^1 + Q^2 \\ \bar{Q}^2 + Q^1 \end{pmatrix}, \quad \tilde{d} = \begin{pmatrix} \bar{Q}^1 - Q^2 \\ \bar{Q}^2 - Q^1 \end{pmatrix}, \quad (49)$$

and \bar{S} can be decomposed. The first term reduces to

$$\bar{\mathcal{L}}_1 = -\frac{1}{2}(\epsilon_{AB}d^Ad^B + \epsilon_{AB}\tilde{d}^A\tilde{d}^B)(\mathcal{V} + \bar{\mathcal{V}}), \quad (50)$$

while the second term is just

$$\bar{\mathcal{L}}_2 = 2Q^a\sigma_{a\dot{a}}^3\bar{Q}^{\dot{a}}\mathcal{V}_3 + 2Q^a\sigma_{a\dot{a}}^\mu\bar{Q}^{\dot{a}}\mathcal{V}_\mu. \quad (51)$$

Remarkably, an explicit calculation shows that both of the terms in $\bar{\mathcal{L}}_2$ are individually $\text{SO}(3, 1)$ invariant. Since there is a unique such invariant bilinear in Q^a and $\bar{Q}^{\dot{a}}$ these two terms in $\bar{\mathcal{L}}_2$ must be proportional, and we can consider just the first. Thus $\bar{\mathcal{L}}_2$ is itself proportional to

$$2Q^a\sigma_{a\dot{a}}^3\bar{Q}^{\dot{a}}\mathcal{V}_3 = -\frac{1}{2}(\epsilon_{AB}d^Ad^B - \epsilon_{AB}\tilde{d}^A\tilde{d}^B)\mathcal{V}_3. \quad (52)$$

This is still not quite in balanced form, since we have two doublets of supercharges. However, it is straightforward to check that

$$\epsilon_{AB}d^Ad^B(\mathcal{V} + \bar{\mathcal{V}}) = \epsilon_{AB}\tilde{d}^A\tilde{d}^B(\mathcal{V} + \bar{\mathcal{V}}), \quad (53)$$

since by (46) the difference may be written as anticommutators of supercharges. Moreover, another explicit calculation shows that

$$\epsilon_{AB}d^Ad^B\mathcal{V}_3 = -\epsilon_{AB}\tilde{d}^A\tilde{d}^B\mathcal{V}_3. \quad (54)$$

Collecting these results we see that $\bar{\mathcal{L}}$ is of balanced form (48) with potential,

$$\begin{aligned} \mathcal{W} = & \frac{i}{4}(\varphi, F_{\alpha\bar{\alpha}}) + \frac{i}{32}\bar{\Omega}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\text{CS}(A)_{\alpha\beta\gamma} - \frac{i}{32}\Omega_{\alpha\beta\gamma}\text{CS}(A)_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \\ & - \frac{1}{16}(\bar{\psi}_{\dot{a}\alpha}, \psi_{a\bar{\alpha}}) + \frac{1}{16}\epsilon^{ab}(\psi_a, \psi_b) - \frac{1}{16}\epsilon^{\dot{a}\dot{b}}(\bar{\psi}_{\dot{a}}, \bar{\psi}_{\dot{b}}), \end{aligned} \quad (55)$$

where we have introduced $\varphi \equiv \phi_3$. Hence the physical supersymmetric Yang–Mills theory differs from this balanced cohomological field theory by the topological term T in (40). Note, however, that T of course depends on the Kähler class.

Balanced theories localise on the critical points of \mathcal{W} . These points correspond to fermions set to zero, and bosons obeying $F_{\alpha\bar{\alpha}} = 0$ and the Bogomol’nyi-type equations

$$\frac{1}{4}\Omega_{\alpha\beta\gamma}F_{\bar{\beta}\bar{\gamma}} + D_{\bar{\alpha}}\varphi = 0 \quad \text{and} \quad \frac{1}{4}\Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}}F_{\beta\gamma} + D_{\alpha}\varphi = 0. \quad (56)$$

For a compact Calabi–Yau 3-fold, these equations reduce to the Kähler–Yang–Mills equations

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0 \quad \text{and} \quad F_{\alpha\bar{\alpha}} = 0, \quad (57)$$

together with the trivial $D_{\alpha}\varphi = D_{\bar{\alpha}}\varphi = 0$.

5. CONCLUSIONS AND OUTLOOK

We have shown that the dimensional reduction of supersymmetric Yang–Mills on a compact Calabi–Yau 3-fold is a cohomological theory which localises on the moduli space of solutions to the Kähler Yang–Mills equations or, by the work of Donaldson and Uhlenbeck–Yau, on the moduli space of stable holomorphic bundles. Observables in this theory correspond to invariants of this moduli space, which generalise the Donaldson invariants in four dimensions. Unlike four dimensions, these are not topological invariants of the Calabi–Yau 3-fold, but a priori only invariants of the $SU(3)$ structure. It follows from the balanced formulation (48) of the theory that $\bar{\mathcal{L}}$ is invariant under infinitesimal deformations of the metric which preserve the Calabi–Yau condition. A similar result was shown in [2] for the cohomological theories on 7- and 8-manifolds of exceptional holonomy.

One direction in which this work may be pursued is to examine the Ansatz of [7] for the effective theory of branes wrapped around supersymmetric cycles in the Calabi–Yau space, here a special lagrangian torus. Considering the embedding of the torus local coordinates of [17] to see how the topological twisting on the torus arises, we note that arguments as in [7] (and [6]) suggest that, for one $U(1)$ case, the resulting path integral on the torus localises on the moduli space $\mathcal{M}_{\text{SL}} \times \mathcal{M}_{\text{Flat}}$, precisely the local description of the mirror [17]. Details will appear elsewhere.

ACKNOWLEDGEMENT

JMF takes pleasure in thanking Bobby Acharya, Chris Hull, Chris Köhl and Bill Spence for conversations on this and other related topics.

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DEPARTMENT OF PHYSICS
 QUEEN MARY AND WESTFIELD COLLEGE
 UNIVERSITY OF LONDON
 LONDON E1 4NS, UK

E-mail address: j.m.figueroa@qmw.ac.uk

DEPARTMENT OF PHYSICS AND MATHEMATICAL PHYSICS
 UNIVERSITY OF ADELAIDE
 ADELAIDE, SA 5005, AUSTRALIA

E-mail address: aimaanpu@physics.adelaide.edu.au

E-mail address: jmccarth@physics.adelaide.edu.au